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On metrics satisfying equation $R_{ij} - \frac{1}{2}Kg_{ij} = T_{ij}$ for constant tensors T [☆]

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Abstract

Necessary and sufficient conditions are given on a constant symmetric tensor T_{ij} on R^n, $n \geq 3$, for which there exist metrics $\bar{g}$, conformal to a pseudo-Euclidean metric g, such that $\bar{R}_{ij} - \frac{1}{2}\bar{K}\bar{g}_{ij} = T_{ij}$, where $\bar{R}_{ij}$ and $\bar{K}$ are the Ricci tensor and the scalar curvature of $\bar{g}$. All solutions $\bar{g}$ are given explicitly and it is shown that there are no complete metrics $\bar{g}$ conformal and nonhomothetic to g.

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In the problem section of the Seminar on Differential Geometry edited by Yau [9], the first problem on Ricci curvature is the following:

> *Find necessary and sufficient conditions on a symmetric tensor T_{ij} on a compact manifold so that one can find a metric g_{ij} to satisfy $R_{ij} - \frac{1}{2}Kg_{ij} = T_{ij}$, whence R_{ij} is the Ricci tensor and K is the scalar curvature of g_{ij}.*

If g_{ij} is the Lorentz metric on a four-dimensional manifold, this is simply the Einstein field equation. Whenever the tensor T represents a physical field such as electromagnetic field perfect fluid type, pure radiation field and vacuum ($T = 0$), the above equation has been studied in several papers, most of them dealing with solutions which are invariant under some symmetry group of the equation (see [5] for details). When the metric g is conformal

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to the Minkowski space–time, then the solutions in the vacuum case are necessarily flat and apparently not known explicitly (see [5]). In the remaining cases, all the solutions conformal to the Minkowski metric are known. We refer to [6] for the pure radiation or the null electromagnetic field, to [2,8] for the non-null electromagnetic field, and finally to [1,8] if T is a perfect fluid. Besides these special results, as far as we know, very little is known about problem (P) with respect to other manifolds, its dimension or the tensor T.

Our purpose in this paper is to solve problem (P) in R^n, $n \geq 3$, for constant symmetric tensors of the form

$$T = \sum_{i,j=1}^{n} \varepsilon_j c_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j \quad \text{with } c_{ij} \in R \text{ and } \varepsilon_j c_{ij} = \varepsilon_i c_{ji} \tag{1}$$

requiring the metric to be conformal to the pseudo-Euclidean metric (R^n, g), $g_{ij} = \delta_{ij}\varepsilon_i$, $\varepsilon_i = \pm 1$, where at least one eigenvalue ε_i is positive. We want to find metrics $\bar{g}$ such that

$$\bar{g} = \frac{1}{\varphi^2} g, \qquad \operatorname{Ric} \bar{g} - \frac{\bar{K}}{2}\bar{g} = T. \tag{2}$$

Before stating our results, we observe that since dimensions higher than 4 are considered in some theories in Physics, it is important to treat this problem in any dimension. Moreover, T being a constant tensor in the standard coordinates of R^n is not a property preserved under a change of coordinates. The requirement of being constant in our paper implies that T is covariantly constant in the standard flat metric g. However, T in general will not be covariantly constant in the metric $\bar{g}$ conformal to g.

We consider the linear functions β_i, $1 \leq i \leq n$ defined for each $x = (x_1, \ldots, x_n) \in R^n$ by $\beta_i(x) = (n-1)(n-2)x_i - (n-3)\sum_{k=1}^{n} x_k$. For a fixed pseudo-Euclidean metric $g_{ij} = \delta_{ij}\varepsilon_i$, we consider the following subsets of R^n:

$$D = \{x \in R^n; \varepsilon_j \beta_j(x) \geq 0 \ \forall j, 1 \leq j \leq n\},$$
$$L = \{x \in R^n; \varepsilon_j \beta_j(x) \leq 0 \ \forall j, 1 \leq j \leq n\},$$
$$\pi_i = \{x \in R^n; \beta_i(x) = 0\}, \quad 1 \leq i \leq n.$$

D and L are nonempty subsets of R^n. With this notation we can now state our results.

Theorem 1. *Let (R^n, g) be a pseudo-Euclidean space and let T be a nondiagonal symmetric tensor as in* (1) *such that $\sum_i c_{ii} \neq 0$. Then there exists a metric $\bar{g}$ solving* (2) *if and only if $c = (c_{11}, \ldots, c_{nn}) \in D \setminus \{\pi_\ell \cup \pi_k\}$ for some $\ell \neq k$ and*

$$c_{ij} = \frac{\varepsilon_j \gamma_i \gamma_j}{(n-1)(n-2)} \sqrt{\varepsilon_i \varepsilon_j \beta_i \beta_j(c)} \quad \forall i \neq j, \tag{3}$$

where $\gamma_j = \pm 1$ for $1 \leq j \leq n$. For any such fixed tensor T, the solutions are $\bar{g} = g/\varphi^2$, where

$$\varphi(x) = k \exp\left(\frac{\delta}{(n-2)\sqrt{(n-1)}} \sum_j \gamma_j \sqrt{\varepsilon_j \beta_j(c)}\, x_j \right), \tag{4}$$

where k is a nonzero constant and $\delta = \pm 1$.

In Theorem 1, for each $c \in D \setminus \{\pi_\ell \cup \pi_k\}$, expressions (3) define at least two and generically 2^{n-1} tensors T.

Theorem 2. *If T be a nondiagonal tensor as in* (1) *such that* $\sum_i c_{ii} = 0$, *then there exists* $\bar{g}$ *solving* (2), *if and only if* $c = (c_{11}, \dots, c_{nn}) \in (D \cup L) \setminus \{\pi_\ell \cup \pi_k\}$ *for some* $\ell \neq k$ *and*

$$c_{ij} = \begin{cases} \varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}} & \forall i \neq j \quad \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\}, \\ -\varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}} & \forall i \neq j \quad \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\}, \end{cases} \tag{5}$$

where $\gamma_j = \pm 1$ *for* $1 \leq j \leq n$. *In this case, for any such fixed tensor T,* $\bar{K} = 0$. *Moreover, the function* φ *is constant if g is the Euclidean metric and otherwise it is given by*

$$\varphi(x) = \begin{cases} k_1 \exp\left(\sum_j h_j(x_j)\right) + k_2 \exp\left(-\sum_j h_j(x_j)\right) & \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\}, \\ k_1 \cos\left(\sum_j h_j(x_j)\right) + k_2 \sin\left(\sum_j h_j(x_j)\right) & \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\} \end{cases} \tag{6}$$

and

$$h_j(x_j) = \begin{cases} \dfrac{\sqrt{\varepsilon_j c_{jj}}\gamma_j x_j}{\sqrt{n-2}} & \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\}, \\ \dfrac{\sqrt{-\varepsilon_j c_{jj}}\gamma_j x_j}{\sqrt{n-2}} & \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\}. \end{cases} \tag{7}$$

The functions φ given in Theorem 2 satisfy $\Delta_g \varphi = \|\nabla_g \varphi\|^2 = 0$.

Theorem 3. *If* $T = \sum_{i=1}^n \varepsilon_i c_{ii}\, \mathrm{d}x_i^2$ *is a nonzero diagonal tensor, then there exists a solution* $\bar{g}$ *of* (2) *if and only if*

$$T = \begin{cases} b\varepsilon_k\, \mathrm{d}x_k^2 & \text{if } n = 3, \\ b\sum_{i\neq k, i=1}^n \varepsilon_i\, \mathrm{d}x_i^2 + \dfrac{n-1}{n-3} b\varepsilon_k\, \mathrm{d}x_k^2 & \text{if } n \geq 4 \end{cases} \tag{8}$$

for some fixed k, $1 \leq k \leq n$, *where b is a real constant such that* $b\varepsilon_k > 0$. *In this case,*

$$\bar{g}_{ij} = \begin{cases} \delta_{ij}\varepsilon_i \exp(a - 2\delta\sqrt{b\varepsilon_k}x_k) & \text{if } n = 3, \\ \delta_{ij}\varepsilon_i \exp\left(a - 2\delta\sqrt{\dfrac{2b\varepsilon_k}{(n-2)(n-3)}}x_k\right) & \text{if } n \geq 4, \end{cases} \tag{9}$$

where $\delta = \pm 1$ *and* $a \in R$.

Theorem 4. *If* $T = 0$, *then there exists a solution* $\bar{g}$ *of* (2) *if and only if*

$$\varphi = \sum_{j=1}^n (A\varepsilon_j x_j^2 + B_j x_j + C_j), \quad \text{where} \quad 4A\sum_j C_j - \sum_j \varepsilon_j B_j^2 = 0 \tag{10}$$

and A, C_j, B_j *are real constants. In this case,* $\bar{K} \equiv 0$, *i.e.* $\mathrm{Ric}\,\bar{g} \equiv 0$.

As a consequence of the above theorems, we obtain the following corollary.

Corollary 5. *Let (R^n, g) be a pseudo-Euclidean space. For any constant symmetric tensor T, there are no complete metrics $\bar{g}$, conformal and nonhomothetic to g, such that* $\mathrm{Ric}\,\bar{g} - \frac{1}{2}\bar{K}\bar{g} = T$.

The techniques used to prove our results are similar to those introduced in [7]. We first recall a well-known result (see for example [4]) that if $\bar{g} = g/\varphi^2$, then

$$\mathrm{Ric}\,\bar{g} - \mathrm{Ric}\,g = \frac{1}{\varphi^2}\{(n-2)\varphi\,\mathrm{Hess}_g(\varphi) + (\varphi\Delta_g\varphi - (n-1)\|\nabla_g\varphi\|^2)g\}.$$

Hence the scalar curvature $\bar{K} = \sum \bar{g}^{ij}\bar{R}_{ij}$ is given by $\bar{K} = (n-1)(2\varphi\Delta_g\varphi - n\|\nabla_g\varphi\|^2)$. Therefore, one proves that solving problem (2) is equivalent to studying the system

$$\varphi_{x_ix_i} = \varepsilon_i\left(\lambda_i\varphi + \frac{\|\nabla_g\varphi\|^2}{2\varphi}\right), 1 \le i \ne j \le n, \qquad \varphi_{x_ix_j} = \frac{\varepsilon_j c_{ij}}{n-2}\varphi, \tag{11}$$

where $\lambda_i = c_{ii}/(n-2) - \sum_\ell c_{\updownarrow\updownarrow}/(n-1)(n-2)$. If φ is a solution of (11), then one can show that

$$c_{ji}\varphi_{x_i} = \frac{\beta_i(c)}{(n-1)(n-2)}\varphi_{x_j} \quad \forall i \ne j. \tag{12}$$

Using (11) and (12), we can prove that if T is nondiagonal and φ is a solution of (11), then $\|\nabla_g\varphi\|^2/2\varphi = -\sum_k\lambda_k\varphi/(n-2)$. It follows from this equation that the diagonal elements of T are such that $c = (c_{11}, \ldots, c_{nn}) \in (D \cup L)\setminus\{\pi_r \cup \pi_\ell\}$ for some pair (r, ℓ), $1 \le r \ne \ell \le n$, and the nondiagonal elements are determined by c as in (3).

If $\sum_i c_{ii} \ne 0$, and $c \in L\setminus\{\pi_r \cup \pi_\ell\}$, then $\varphi = 0$, hence we conclude that in this case, $c \in D\setminus\{\pi_r \cup \pi_\ell\}$ and φ is given by (4). If $\sum_i c_{ii} = 0$, then (11) reduces to $\varphi_{x_ix_i} = \varepsilon_i c_{ii}\varphi/(n-2)$ and $\varphi_{x_ix_j} = \varepsilon_j c_{ij}\varphi/(n-2)$ for $i \ne j$. Therefore, one can show that φ is given by (6) and the elements of T satisfy (5).

In order to prove Theorem 3, we observe that if T is a nonzero diagonal tensor, it follows from (12) that φ is not constant and $0 = \beta_i(c)\varphi_{x_j}\forall i \ne j$. Let k be such that $\varphi_{x_k} \ne 0$. If $n \ge 4$, then for all $i \ne k$, $c_{ii} = b$, where $b \ne 0$ is a real constant and $\beta_i = 0$. We conclude that $c_{kk} = (n-1)b/(n-3)$. If $n = 3$, then $c_{ii} = 0$ for all $i \ne k$ and $c_{kk} = b \ne 0$. In both cases, φ depends only on x_k. Therefore, T is given by (8) and the system (11) reduces to ordinary differential equations whose solution provides $\bar{g}$ as in (9).

The proof of Theorem 4 follows immediately from the fact that φ satisfies the system of Eq. (11), where $\lambda_i = 0$ for all i.

The converse of Theorems 1–4 follows from a straightforward computation.

For each fixed tensor T as in Theorem 1 or 3, there exists two semi-Riemannian metrics (given by $\delta = \pm 1$) in the same conformal class which have pointwise the same Ricci tensor. Since they are not homothetic to each other, it follows from the results of [3,4] that they are not complete. A similar argument applies to the metrics obtained in Theorem 2 when $c \in D\setminus\{\pi_\ell \cup \pi_k\}$. In the remaining cases, the metric $\bar{g} = g/\varphi^2$ has singularity points. This completes the proof of Corollary 5.

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